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OPTIMAL DIGITAL REDESIGN OF CONTINUOUS-TIME CONTROLLERS

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Abstract—This paper proposes a new optimal digital redesign technique for finding a dynamic digital control law from the available analog counterpart and simultaneously minimizing a quadratic performance index. The proposed technique can be applied to a system with a more general class of reference inputs, and the developed digital regulator can be implemented using low cost microcomputers.

1. INTRODUCTION

Many practical dynamic systems are described by continuous-time state equations for which a state-feedback gain and a forward gain are designed based upon some specific desired goals. Advances in digital control theory and industrial electronics have made a dramatic extension in the possibilities of replacing these analog controllers by the equivalent digital controllers so that they can be implemented using high performance, low cost microprocessors and associated microelectronics. The conversion of the designed continuous-time controller (analog controller) to an equivalent discrete-time controller (digital controller), so that the responses of the redesigned equivalent digital system closely match those of the original analog system for the same input and initial conditions, is a digital redesign problem [1]. The digital redesign problem can be described as follows.

Consider the linear controllable continuous-time system described by

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t); \quad x_c(0) = x_0, \tag{1}$$

where $x_c(t)$ and $u_c(t)$ are an $n \times 1$ state vector and an $m \times 1$ input vector, respectively, and A and B are constant matrices of appropriate dimensions. Let the state-feedback control law be

$$u_c(t) = -K_c x_c(t) + E_c r(t), \tag{2}$$

where K_c is an $m \times n$ feedback gain, E_c is an $m \times m$ forward gain, and $r(t)$ is an $m \times 1$ reference input. The resulting closed-loop system becomes

$$\dot{x}_c(t) = (A - BK_c)\,x_c(t) + BE_c r(t); \quad x_c(0) = x_0. \tag{3}$$

Let the state equation of a continuous-time system which contains the same system matrix A and input matrix B of the system in (1), with a different input, be represented by

$$\dot{x}_d(t) = Ax_d(t) + Bu_d(t); \quad x_d(0) = x_0, \tag{4a}$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $u_d(t)$ is an $m \times 1$ piecewise-constant input function,

$$u_d(t) = u_d(kT) \qquad \text{for} \quad kT \le t < (k+1)T \tag{4b}$$

and T is the sampling period. A zero-order hold is utilized in (4). The solution of the state equation in (4) is

$$x_d(t) = e^{A(t-kT)}x_d(kT) + \int_{kT}^{t} e^{A(t-\lambda)}B\,d\lambda\, u_d(kT) \quad \text{for} \quad kT \le t \le kT+T. \tag{5}$$

For $t = kT + T$, the equivalent discrete-time model of the continuous-time system in (4) can be written as

$$x_d(kT+T) = Gx_d(kT) + Hu_d(kT); \quad x_d(0) = x_0, \tag{6a}$$

where

$$G \triangleq e^{AT} \quad \text{and} \quad H \triangleq \int_0^T e^{A\lambda}B\,d\lambda = [G - I_n]A^{-1}B. \tag{6b}$$

Let the state-feedback control law for the system in (4) be

$$u_d(kT) = -K_d x_d(kT) + E_d r(kT), \tag{7}$$

where K_d is an $m \times n$ digital feedback gain, E_d is an $m \times m$ digital forward gain, and $r(kT)$ is an $m \times 1$ discrete-time reference input. The resulting closed-loop system becomes

$$\dot{x}_d(t) = Ax_d(t) - BK_d x_d(kT) + BE_d r(kT); \quad x_d(0) = x_0 \quad \text{for} \quad kT \le t < (k+1)T \tag{8}$$

Now, the static digital redesign problem reduces to finding the digital constant state-feedback gain K_d and forward gain E_d in (7) from the continuous state-feedback gain K_c and forward gain E_c in (2) so that the states of the digital model in (8) are approximately equal to the states of the analog system in (3) for $x_c(0) = x_d(0)$ and the same reference input.

In Kuo's work [1], a discrete-state matching method was proposed to solve the static digital redesign problem and successfully applied to a simplified one-axis skylab satellite system [1]. In their work [1], they have assumed that the continuous-time reference input $r(t)$ in (2) can be closely approximated by the piecewise-constant input $r(kT)$ in (8) and, therefore, the continuous-time state $x_c(t)$ in (3) can be approximated by the continuous-time state $x_d(t)$ in (8) at each sampling instant, $t = kT$, with a sufficiently small sampling period T. The values of $x_c(t)$ in (3) and $x_d(t)$ in (8) between each sampling instant are not considered in [1]. In this paper, a new optimal digital redesign technique is proposed for finding a dynamic digital control law, instead of the static digital control law as shown in (7), from the available analog control law in (2) for a continuous-time reference input $r(t)$. It is optimal in the sense that the quadratic performance index of the errors between $x_c(t)$ in (3) and the digital redesigned state, controlled by a dynamic digital control law, is minimized.

2. OPTIMAL DIGITAL REDESIGN

Consider the dynamic system described as in (3) and (4) with $x_c(0) = x_d(0)$. Let the quadratic cost function be

$$J = \frac{1}{2}\int_0^{\infty} [x_d(t) - x_c(t)]^T Q [x_d(t) - x_c(t)]\,dt, \tag{9}$$

where $Q \in \mathcal{R}^{n\times n}$ is a positive definite symmetric weighting matrix, $x_c(t)$ is the state of the system in (3), and $x_d(t)$ is the state of the system in (4) with the $u_d(t)$ to be redesigned. The objective is to find a dynamic digital control law for the system in (6a) such that J in (9) is minimized. The above optimization problem is slightly different from an optimal state tracking problem [2] in the sense that the states of interest in (9) are those of the dynamic systems in (3) and (4) which involve the same system matrix A and input matrix B, but with different input functions.

An alternative expression of J in (9) is

$$J = \sum_{k=0}^{\infty} \frac{1}{2} \int_{kT}^{kT+T} [x_d(t) - x_c(t)]^T Q [x_d(t) - x_c(t)]\, dt \triangleq \sum_{k=0}^{\infty} J_k, \tag{10a}$$

where

$$J_k \triangleq \frac{1}{2} \int_{kT}^{kT+T} [x_d(t) - x_c(t)]^T Q [x_d(t) - x_c(t)]\, dt. \tag{10b}$$

Assume that the continuous-time reference input $r(t)$ in (3) can be realized via zero-input state equations (see Appendix A) as in the following equations:

$$\dot{y}_r(t) = A_r y_r(t); \quad y_r(0) = y_0 \tag{11a}$$

$$r(t) = C_r y_r(t), \tag{11b}$$

where $y_r(t)$ is a $p_r \times 1$ state vector, $r(t)$ in (11b) is an $m \times 1$ output vector (the reference input of the system in (3)), and A_r and C_r are constant matrices of appropriate dimensions. Combining the state equations in (3) and (11) leads to

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{y}_r(t) \end{bmatrix} = \begin{bmatrix} A_c & BE_cC_r \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x_c(t) \\ y_r(t) \end{bmatrix}; \quad \begin{bmatrix} x_c(0) \\ y_r(0) \end{bmatrix} \in \mathcal{R}^{n_1 \times 1} \tag{12a}$$

or

$$\dot{q}(t) = A_1 q(t); \qquad q(0) = q_0, \tag{12b}$$

where $n_1 \triangleq n + p_r$, $A_c \triangleq A - BK_c$, and

$$A_1 \triangleq \begin{bmatrix} A_c & BE_cC_r \\ 0 & A_r \end{bmatrix} \in \mathcal{R}^{n_1 \times n_1}, \quad q(t) \triangleq \begin{bmatrix} x_c(t) \\ y_r(t) \end{bmatrix} \in \mathcal{R}^{n_1 \times 1}.$$

The solution of the state equation in (12) is given by

$$q(t) = e^{A_1(t-kT)} q(kT) \quad \text{for} \quad kT \le t \le kT + T. \tag{13a}$$

The equivalent discrete-time model of the system in (13a) is

$$q(kT + T) = G_1 q(kT); \quad q(0) = q_0, \tag{13b}$$

where

$$G_1 \triangleq e^{A_1 T} \triangleq \begin{bmatrix} G_c & H_c \\ 0 & G_r \end{bmatrix} \in \mathcal{R}^{n_1 \times n_1}, \quad q(kT) = \begin{bmatrix} x_c(kT) \\ y_r(kT) \end{bmatrix} \in \mathcal{R}^{n_1 \times 1},$$

with $G_c \triangleq e^{A_cT}$, $G_r \triangleq e^{A_rT}$, and $H_c \triangleq G_c \int_0^T e^{-A_c\lambda} BE_cC_r e^{A_r\lambda}\, d\lambda$ (Note that H_c can be solved using the method developed in Appendix B). Also, by combining the dynamic systems in (4) and (12), we obtain

$$\begin{bmatrix} \dot{x}_d(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x_d(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_d(kT) \tag{14a}$$

The solution of the state equation in (14a), obtained by combining the solutions in (5) and (13a), is

$$\begin{bmatrix} x_d(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} e^{A(t-kT)} & 0 \\ 0 & e^{A_1(t-kT)} \end{bmatrix} \begin{bmatrix} x_d(kT) \\ q(kT) \end{bmatrix} + \begin{bmatrix} \int_{kT}^{t} e^{A(t-\lambda)} B\, d\lambda \\ 0 \end{bmatrix} u_d(kT) \quad \text{for} \quad kT \le t \le kT + T \tag{14b}$$

For $t = kT + T$, the equivalent discrete-time model of the augmented system in (14) becomes

$$z(kT+T) = \hat{G}z(kT) + \hat{H}u_d(kT); \quad z(0) = z_0, \tag{15}$$

where

$$z(kT) = [x_d^T(kT), q^T(kT)]^T \in \mathcal{R}^{(n+n_1)\times 1}, \quad z_0 = [x_0^T, q_0^T]^T$$
$$\hat{G} \triangleq \text{ block diag } [G, G_1] \in \mathcal{R}^{(n+n_1)\times(n+n_1)} \text{ with } \quad G = e^{AT} \quad \text{and} \quad G_1 = e^{A_1T}$$
$$\text{and} \quad \hat{H} \triangleq [H^T, 0]^T \in \mathcal{R}^{(n+n_1)\times m} \quad \text{with} \quad H = \int_0^T e^{A\lambda}B\, d\lambda = [G - I_n]A^{-1}B.$$

Note that the augmented system in (15) contains the reference subsystem in (11), whereas the cost function in (9) does not include the state of the reference subsystem in (11). To include the state of the reference subsystem in (11) into the cost function in (9), we modify the cost function in (10) as follows:

$$\begin{aligned} J_k &= \frac{1}{2}\int_{kT}^{kT+T} [x_d^T(t), x_c^T(t)] \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix} \begin{bmatrix} x_d(t) \\ x_c(t) \end{bmatrix} dt \\ &= \frac{1}{2}\int_{kT}^{kT+T} z^T(t)\,\tilde{Q}z(t)\, dt, \end{aligned} \tag{16}$$

where

$$z(t) \triangleq [x_d^T(t), q^T(t)]^T = [x_d^T(t), x_c^T(t), y_r^T(t)]^T \in \mathcal{R}^{(n+n_1)\times 1}$$
$$\tilde{Q} \triangleq \begin{bmatrix} Q & -Q & 0 \\ -Q & Q & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{R}^{(n+n_1)\times(n+n_1)}.$$

Substituting (14b) into (16) and making some algebraic simplifications results in

$$J_k = \frac{1}{2}z^T(kT)\,\hat{Q}z(kT) + z^T(kT)\,Mu_d(kT) + \frac{1}{2}u_d^T(kT)\,Ru_d(kT), \tag{17a}$$

where

$$\hat{Q} \triangleq \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}^T & \hat{Q}_{22} \end{bmatrix} \in \mathcal{R}^{(n+n_1)\times(n+n_1)}, \tag{17b}$$

with

$$\hat{Q}_{11} \triangleq \int_0^T e^{A^Tt}Qe^{At}\, dt \in \mathcal{R}^{n\times n}$$
$$\hat{Q}_{12} \triangleq \int_0^T e^{A^Tt}[-Q, 0]e^{A_1t}\, dt \in \mathcal{R}^{n\times n_1}$$
$$\hat{Q}_{22} \triangleq \int_0^T e^{A_1^Tt}\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} e^{A_1t}\, dt \in \mathcal{R}^{n_1\times n_1}$$

and

$$M \triangleq \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \in \mathcal{R}^{(n+n_1)\times m}, \tag{17c}$$

with

$$M_1 \triangleq \int_0^T \{e^{A^Tt}Q\int_0^t e^{A\lambda}B\, d\lambda\}\, dt \in \mathcal{R}^{n\times m}$$
$$M_2 \triangleq \int_0^T \{e^{A_1^Tt}[-Q, 0]^T\int_0^t e^{A\lambda}B\, d\lambda\}\, dt \in \mathcal{R}^{n_1\times m}$$

and

$$R \triangleq \int_0^T \left\{ \left[\int_0^t e^{A\lambda} B\, d\lambda\right]^T Q \left[\int_0^t e^{A\lambda} B\, d\lambda\right]\right\} dt \in \mathcal{R}^{m\times m}. \tag{17d}$$

If the matrices A and A_1 satisfy certain conditions (see Appendix B), the weighting matrices $\hat{Q}$, M, and R can be solved from a set of Lyapunov equations. Thus, the quadratic cost function in (10) can be rewritten as

$$J = \sum_{k=0}^{\infty} \left[\frac{1}{2} z^T(kT)\hat{Q} z(kT) + z^T(kT) M u_d(kT) + \frac{1}{2} u_d^T(kT) R u_d(kT)\right] \tag{18}$$

Now, we can easily identify that the cost function in (18) and the dynamic equation in (15) constitute a standard discrete-time optimal regulator problem [1,2]. The optimal control law is given [1,2] by

$$u_d(kT) = -(R + \hat{H}^T P \hat{H})^{-1} (\hat{H}^T P \hat{G} + M^T) z(kT), \tag{19a}$$

where $P \in \mathcal{R}^{(n+n_1)\times(n+n_1)}$ is the positive definite symmetric solution of the discrete-time Riccati equation:

$$P = \hat{G}^T P \hat{G} + \hat{Q} - (\hat{G}^T P \hat{H} + M)(R + \hat{H}^T P \hat{H})^{-1} (\hat{G}^T P \hat{H} + M)^T. \tag{19b}$$

Since the adjoint system in (15) is not completely controllable, it is not always possible to find a positive semidefinite symmetric matrix P from (19b). However, for a stable subsystem matrix G_1, there exists a positive definite symmetric matrix P [3] which can be solved as follows.

Define the matrix P as

$$P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \tag{20a}$$

where $P_{11} \in \mathcal{R}^{n\times n}$, $P_{12} \in \mathcal{R}^{n\times n_1}$, and $P_{22} \in \mathcal{R}^{n_1\times n_1}$. The Riccati equation in (19b) can be partitioned into separate equations for P_{11}, P_{12}, and P_{22}:

$$P_{11} = G^T P_{11} G + \hat{Q}_{11} - (G^T P_{11} H + M_1)(R + H^T P_{11} H)^{-1} (G^T P_{11} H + M_1)^T \tag{20b}$$

$$P_{12} = G^T P_{12} G_1 + \hat{Q}_{12} - (G^T P_{11} H + M_1)(R + H^T P_{11} H)^{-1} (G_1^T P_{12}^T H + M_2)^T \tag{20c}$$

$$P_{22} = G_1^T P_{22} G_1 + \hat{Q}_{22} - (G_1^T P_{12}^T H + M_2)(R + H^T P_{11} H)^{-1} (G_1^T P_{12}^T H + M_2)^T. \tag{20d}$$

Equation (20b) is a discrete-time algebraic Riccati equation and can be solved using the eigenvalue-eigenvector approach [4] or sign algorithm [5]. Once P_{11} has been found, it is substituted into (20c), which can be rearranged into the following Lyapunov equation:

$$\begin{aligned}&[G - H(R + H^T P_{11} H)^{-1} (H^T P_{11} G + M_1^T)]^T P_{12} - P_{12} G_1^{-1} \\ &\qquad + [\hat{Q}_{12} - (G^T P_{11} H + M_1)(R + H^T P_{11} H)^{-1} M_2^T] G_1^{-1} = 0.\end{aligned} \tag{21}$$

Equation (21) can be solved using a matrix direct-product method [6]. The desired optimal digital control law in (19a) becomes

$$u_d(kT) = -K_d x_d(kT) + K_q q(kT), \tag{22}$$

where

$$K_d = (R + H^T P_{11} H)^{-1} (H^T P_{11} G + M_1^T)$$
$$K_q = -(R + H^T P_{11} H)^{-1} (H^T P_{12} G_1 + M_2^T).$$

$q(kT)$ in (22) is generated from the dynamic system in (12) or (13) as shown in the following equation:

$$q(kT+T) = \begin{bmatrix} x_c(kT+T) \\ y_r(kT+T) \end{bmatrix} = \begin{bmatrix} G_c & H_c \\ 0 & G_r \end{bmatrix} \begin{bmatrix} x_c(kT) \\ y_r(kT) \end{bmatrix}; \quad \begin{bmatrix} x_c(0) \\ y_r(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{23}$$

We decompose the dynamic gain K_q in (22) as $K_q = [\hat{K}_c, \hat{K}_r]$, where $\hat{K}_c \in \mathcal{R}^{m\times n}$ and $\hat{K}_r \in \mathcal{R}^{m\times p_r}$. Hence, the desired optimal dynamic digital control law in (22) can be rewritten as

$$u_d(kT) = -K_d x_d(kT) + \hat{K}_c x_c(kT) + \hat{K}_r y_r(kT), \tag{24a}$$

where

$$x_c(kT) = G_c^k\, x_d(0) + \sum_{i=0}^{k-1} G_c^{k-i-1}\, H_c\, y_r(iT) \tag{24b}$$

and

$$y_r(kT) = G_r^k\, y_r(0). \tag{24c}$$

Thus, the digital redesigned system using the optimal dynamic digital controller in (24) becomes

$$\dot{x}_d(t) = Ax_d(t) - BK_d\, x_d(kT) + B\hat{K}_c\, x_c(kT) + B\hat{K}_r\, y_r(kT); \quad x_d(0) = x_0 \tag{25}$$

The digital redesigned closed-loop system is shown in Figure 1.

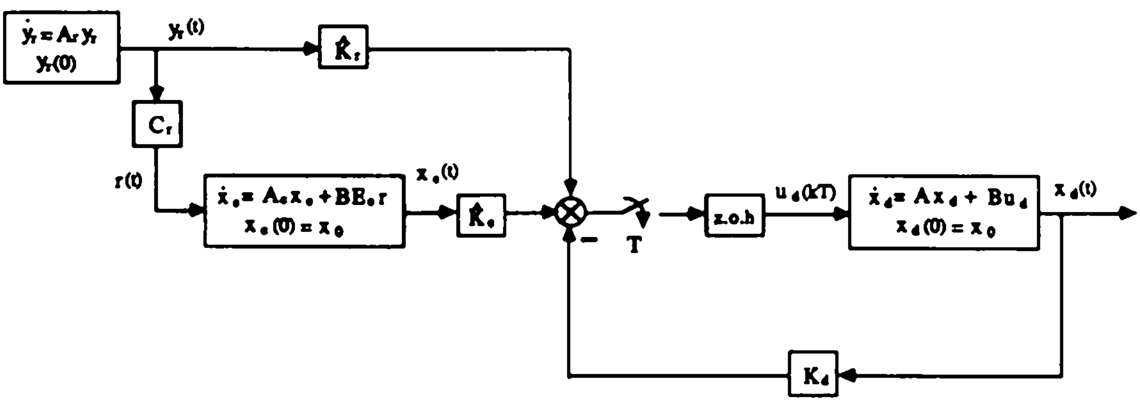

Figure 1. The digital redesigned closed-loop system.

If $y_r(t)$ in (11) is measurable, or the initial vector $y_r(0)$ is available, the control law in (24a) can be realized using a microcomputer. However, in practice, it is quite possible that only an incoming signal $r(t)$ is available. In this case, an estimator can be constructed with $r(t)$ as an input, and the estimated state $\hat{y}_r(t)$ of $y_r(t)$ as an output [2] provided that the pair $[A_r, C_r]$ are observable.

When $r(t)$ in (11) is a step function, then $C_r = I_m$, $y_r(t) = r(t)$, and $y_r(kT) = r(kT)$. The optimal dynamic digital control law in (24) reduces to

$$\begin{aligned} u_d(kT) &= -K_d\, x_d(kT) + \hat{K}_c\, x_c(kT) + \hat{K}_r\, r(kT) \\ &= -K_d x_d(kT) + \hat{K}_c G_c^k\, x_d(0) + [\hat{K}_c \sum_{i=0}^{k-1} G_c^{k-i-1} H_c + \hat{K}_r]\, r(kT). \end{aligned} \tag{26}$$

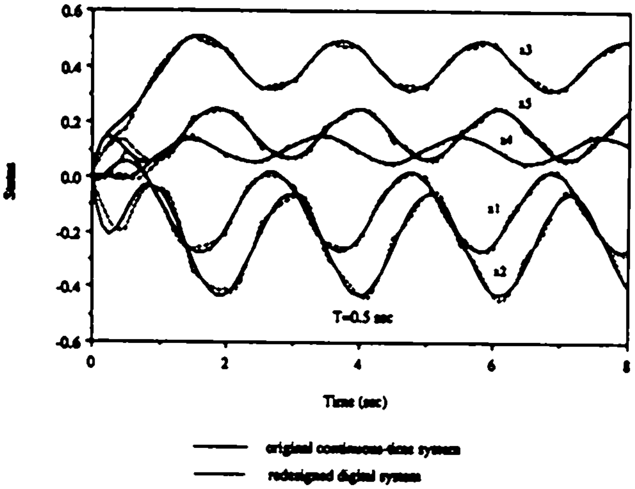

Figure 2. The state responses of the original closed-loop system in (3) and the digital redesigned closed-loop system in (25).

3. ILLUSTRATIVE EXAMPLE

Consider an unstable system in (1) with

$$A = \begin{bmatrix} 0.809 & -2.060 & 0.325 & 0.465 & 0.895 \\ 6.667 & 0.200 & 1.333 & 0.000 & 0.667 \\ -1.291 & 0.458 & -1.072 & -2.326 & -0.199 \\ -0.324 & 0.824 & 1.670 & -1.186 & -0.358 \\ -3.509 & -4.316 & -0.702 & 0.000 & -8.351 \end{bmatrix} \tag{27a}$$

$$B = \begin{bmatrix} 0.955 & -0.379 \\ -1.667 & -1.667 \\ -0.212 & 1.195 \\ 0.618 & 0.052 \\ 0.877 & 1.403 \end{bmatrix}; \quad x_c(0) = 0 \tag{27b}$$

and the eigenvalues of A are $\sigma(A) = \{0.2 \pm j4.0, -1.0 \pm j2.0, -8.0\}$.

Using the optimal pole-placement method proposed in [7], the optimal state feedback gain K_c in (2) is found as

$$K_c = \begin{bmatrix} 7.871 & -0.563 & 3.255 & -0.137 & 0.754 \\ 1.625 & -1.247 & 1.297 & -1.003 & 0.182 \end{bmatrix} \tag{28}$$

Utilizing the feedback gain K_c, the eigenvalues of the closed-loop system in (3) are placed within the common region of an open sector (with a sector angle $\pm 45°$ from the negative real axis) and the left-hand side of a -1.1 vertical line on the negative real axis in the complex s-plane, and $\sigma(A - BK_c) = \{-4.6789 \pm j4.6518, -1.8983 \pm j1.898, -8.0\}$.

Assume $E_c = I_2$ in (2), and let that the reference input $r(t)$ in (2) contain a sine function ($\sin(\omega t)$) with an angular frequency $\omega = 3.0$ and a unit-step function, i.e.,

$$r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} \sin(3.0t) \\ 1 \end{bmatrix}; \quad t \geq 0 \tag{29}$$

The reference input $r(t)$ can be represented by a zero-input state equation in (11) with

$$A_r = \begin{bmatrix} 0.0 & 3.0 & 0.0 \\ -3.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{30a}$$

$$C_r = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{30b}$$

and an initial vector $y_r(0) = [0.0 \quad 1.0 \quad 1.0]^T$.

Using the method proposed in this paper, we obtain the dynamic digital control law in (24) with a sampling period $T = 0.5\,(sec.)$ as

$$u_d(kT) = -K_d\, x_d(kT) + \hat{K}_c\, x_c(kT) + \hat{K}_r\, y_r(kT), \tag{31}$$

where

$$K_d = \begin{bmatrix} 0.7940 & -1.0970 & -0.2206 & 0.3500 & 0.0574 \\ -2.2280 & -0.2333 & 0.0002 & -0.7134 & -0.1575 \end{bmatrix} \tag{32a}$$

$$\hat{K}_c = \begin{bmatrix} -0.4472 & -0.2190 & -0.7267 & 0.2998 & -0.0587 \\ -0.0501 & 0.0972 & 0.5231 & -0.2748 & 0.0399 \end{bmatrix} \tag{32b}$$

$$\hat{K}_r = \begin{bmatrix} 0.2112 & 0.2314 & -0.2938 \\ -0.1482 & -0.0498 & 0.5082 \end{bmatrix} \tag{32c}$$

The simulation results of the closed-loop systems in (3) and (25) are shown in Figure 2 for both $x_c(t)$ in (3) and $x_d(t)$ in (25), and those of the controls $u_c(t)$ in (2) and $u_d(t)$ in (24) are shown in Figure 3. The simulation results illustrate $x_d(t)$ is very close to $x_c(t)$ even with a rather larger sampling period (considering the dynamics of the given system and the frequency of the reference input).

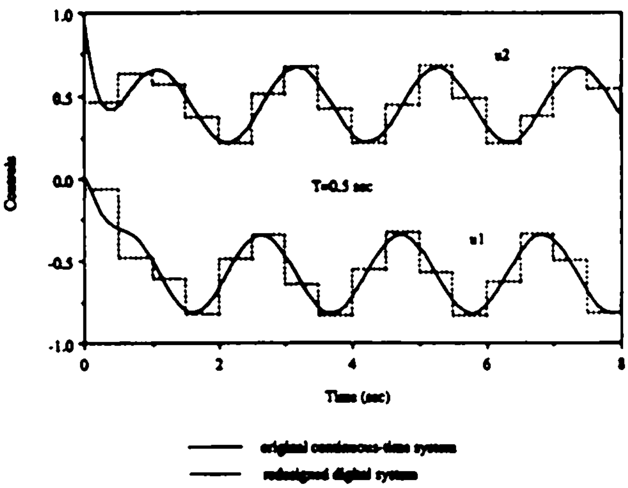

Figure 3. Controls $u_c(t)$ in (2) and $u_d(t)$ in (24).

4. CONCLUSION

A new optimal digital redesign technique has been developed for finding a dynamic digital control law in (24) from the given analog counterpart in (2) and simultaneously minimizing a quadratic performance index in (9). The proposed technique is based on an augmented digital system, constructed from the reference model, the original closed-loop system and the digitally controlled closed-loop system, and the minimization of a quadratic performance index, defined as the difference between the states of the original closed-loop system and the digitally controlled closed-loop system. The weighting matrices of the performance index are determined by solving a set of Lyapunov equations, and the discrete optimal regulator is obtained by solving a low-dimensional Riccati equation and a Lyapunov equation. The computation of the state-feedback and forward gains is straight forward resulting in the digital control law. An illustrative example has been presented to demonstrate the effectiveness of the proposed method. The developed, dynamic, digital, redesigned control law enables an optimally close matching of the states of the digital, redesigned, closed-loop system, as compared to the states of the original closed-loop system, and it can be implemented using low cost microcomputers. The proposed technique can be applied to a system with a more general class of reference inputs having a relatively large sampling period.

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APPENDIX A

Let an $m \times 1$ output rational function $R(s)$, which is the product of a transfer function matrix and an input function, be represented by an irreducible left matrix fraction description [4] as

$$R(s) = [I_m s^p + D_1 s^{p-1} + \ldots + D_p]^{-1}[N_1 s^{p-1} + \ldots + N_p], \tag{A.1}$$

where $D_i \in \mathcal{R}^{m\times m}$ and $N_i \in \mathcal{R}^{m\times m}$ for $i = 1, 2, \ldots, p$.

The left matrix fraction description can be realized by the following zero-input state equation:

$$\dot{y}_r(t) = A_r\, y_r(t); \quad y_r(0) \tag{A.2}$$

$$r(t) = C_r\, y_r(t), \tag{A.3}$$

where $y_r(t) \in \mathcal{R}^{pm\times 1}$, $r(t) \in \mathcal{R}^{m\times 1}$

$$A_r = \begin{bmatrix} 0_m & I_m & 0_m & \ldots & 0_m \\ 0_m & 0_m & I_m & \ldots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -D_p & -D_{p-1} & -D_{p-2} & \ldots & -D_1 \end{bmatrix}, \quad C_r^T = \begin{bmatrix} I_m \\ 0_m \\ \vdots \\ 0_m \end{bmatrix}$$

and

$$y_r(0) = \begin{bmatrix} I_m & 0_m & 0_m & \ldots & 0_m & 0_m \\ D_1 & I_m & 0_m & \ldots & 0_m & 0_m \\ D_2 & D_1 & I_m & \ldots & 0_m & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{p-1} & D_{p-2} & D_{p-3} & \ldots & D_1 & I_m \end{bmatrix}^{-1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ \vdots \\ N_p \end{bmatrix}.$$

The above result can be obtained by following the method shown in [8].

APPENDIX B [9]

Some useful formulas for computing the matrices $\hat{Q}_{11}$, $\hat{Q}_{12}$, M_1, M_2, and R are given as follows.

Let the matrix $\hat{Q}_{11}$ be defined as

$$\hat{Q}_{11} \triangleq \int_0^T e^{A^T t} Q e^{At}\, dt = \int_0^T F_Q(t)\, dt, \tag{B.1a}$$

where

$$F_Q(t) \triangleq e^{A^T t} Q e^{At}. \tag{B.1b}$$

Taking the derivative of (B.1b) with respect to t gives

$$\dot{F}_Q(t) = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A = A^T F_Q(t) + F_Q(t)\, A. \tag{B.2}$$

Integrating (B.2) on both sides from 0 to T yields

$$\int_0^T \dot{F}_Q(t)\,dt = F_Q(t)|_0^T = e^{A^T T} Q e^{AT} - Q$$
$$= A^T \int_0^T F_Q(t)\,dt + \int_0^T F_Q(t)\,dt\,A.$$

Thus,

$$A^T \hat{Q}_{11} + \hat{Q}_{11} A = G^T Q G - Q, \tag{B.3}$$

where $G \triangleq e^{AT}$.

When A is nonsingular, the unique solution $\hat{Q}_{11}$ can be solved from the Lyapunov equation in (B.3) using the matrix direct-product method [6].

Let $\hat{Q}_{12}$ be defined as

$$\hat{Q}_{12} \triangleq \int_0^T e^{A^T t} \hat{Q} e^{A_1 t}\,dt, \tag{B.4}$$

where $\hat{Q} \triangleq [-Q, 0] \in \mathcal{R}^{n \times n_1}$. Similar to the above derivation, if A and A_1 are nonsingular and $\sigma_i(A) + \sigma_j(A_1) \neq 0$ for all i, j, where $\sigma(\cdot)$ denotes the eigenspectrum of $(\cdot)$, then the unique solution $\hat{Q}_{12}$ can be obtained from the following Lyapunov equation:

$$A^T \hat{Q}_{12} + \hat{Q}_{12} A_1 = G^T \hat{Q} G_1 - \hat{Q}, \tag{B.5}$$

where $G \triangleq e^{AT}$ and $G_1 \triangleq e^{A_1 T}$.

Let M_1 be defined as

$$M_1 \triangleq \int_0^T F_M(t)\,dt, \tag{B.6a}$$

where

$$F_M(t) \triangleq e^{A^T t} Q \int_0^t e^{A\lambda} B\,d\lambda. \tag{B.6b}$$

Carrying out the differentiation of (B.6b) with respect to t, we obtain

$$\dot{F}_M(t) = A^T e^{A^T t} Q \int_0^t e^{A\lambda} B\,d\lambda + e^{A^T t} Q e^{At} B$$
$$= A^T F_M(t) + F_Q(t) B. \tag{B.7}$$

Integrating both sides of (B.7) from 0 to T gives

$$F_M(t)|_0^T = e^{A^T T} Q \int_0^T e^{A\lambda} B\,d\lambda = G^T Q H$$
$$= A^T \int_0^T F_M(t)\,dt + \int_0^T F_Q(t)\,dt\,B = A^T M_1 + \hat{Q}_{11} B,$$

where $H \triangleq \int_0^T e^{A\lambda} B\,d\lambda$. Thus,

$$A^T M_1 + \hat{Q}_{11} B = G^T Q H. \tag{B.8}$$

If A is nonsingular, then

$$M_1 = (A^T)^{-1} [G^T Q H - \hat{Q}_{11} B]. \tag{B.9}$$

Define

$$M_2 \triangleq \int_0^T \left\{ e^{A_1^T t} \hat{Q}^T \left[\int_0^t e^{A\lambda} B\,d\lambda \right] \right\} dt.$$

Similar to the derivations of (B.6a) through (B.9), if A_1 is nonsingular, M_2 can be found as

$$M_2 = (A_1^T)^{-1} [G_1^T \hat{Q}^T H - \hat{Q}_{12}^T B]$$

Let R be defined as

$$R \triangleq \int_0^T \left\{ \left[\int_0^t e^{A\lambda} B\,d\lambda \right]^T Q \left[\int_0^t e^{A\lambda} B\,d\lambda \right] \right\} dt \tag{B.10}$$

If A is nonsingular, then

$$\int_0^t e^{A\lambda} B\,d\lambda = [e^{At} - I_n] A^{-1} B \tag{B.11}$$

Substituting (B.11) into (B.10), we have

$$R = (A^{-1}B)^T \int_0^T \left[e^{A^T t} Q e^{At} - e^{A^T t} Q - Q e^{At} + Q \right] dt \, (A^{-1}B)$$
$$= (A^{-1}B)^T [\hat{Q}_{11} - (G - I_n)^T (A^T)^{-1} Q - Q(G - I_n) A^{-1} + QT](A^{-1}B) \quad \text{(B.12)}$$

If A and/or A_1 are singular, the matrices $\hat{Q}_{11}$, $\hat{Q}_{12}$, M_1, M_2, and R can be computed by any numerical integration method.